Part II Classes of Approximating Functions

- Polynomials in R^1

Given a scalar-valued function $f(x)$ defined on $[a,b]$ how to approximate $f(x)$ by a polynomial of degree n?

Let $\mathcal{P}_n = \{ p(x) \mid p(x)$ is a poly of degree less than or equal to $n \}$

$$= \text{span}\{1, x, \cdots, x^n\}$$

$$= \text{span}\{L_0(x), L_1(x), \cdots, L_n(x)\}$$

where $\{L_i(x)\}_{i=0}^{n}$ is an orthonormal basis w.r.t. an inner product.

(1) interpolation

Assume that $\Delta: a \le x_0 \le x_1 \le \cdots \le x_n \le b$ is a partition of the interval $I = [a,b]$.

Find $p_n(x) \in \mathcal{P}_n$ s.t.

(I) $p_n(x_i) = f(x_i)$ for $i = 0, 1, \cdots, n$.

Theorem Assume that $\{x_i\}_{i=0}^{n}$ are distinct, i.e., $x_i \neq x_j$ if $i \neq j$.

Then (I) has a unique solution

$$p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x), \quad \text{where } l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}$$

is the Lagrange nodal basis function, i.e., $l_i(x_j) = \delta_{ij}$.

Moreover, there exists a $\xi \in (a,b)$ s.t.

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (x-x_j).$$

(2) Least-squares Approximation

$$(LS)\ \begin{cases} \text{Find } p_n \in \mathcal{P}_n \text{ s.t.} \\ \|f-p_n\| = \min\limits_{g\in\mathcal{P}_n} \|f-g\| \end{cases} \iff \|f-p_n\| \le \|f-g\| \ \forall g \in \mathcal{P}_n$$

where $\|g\| = \langle g, g\rangle^{\frac{1}{2}}$ is the induced norm of the inner product.

Theorem (LS) has a unique solution

$$p_n(x) = \sum_{i=0}^{n} \langle f, L_i\rangle L_i(x).$$

Assume that there exists a orthonormal basis $\{L_i(x)\}_{i=0}^{\infty}$.

Then we have

$$f(x) - p_n(x) = \sum_{i=n+1}^{\infty} \langle f, L_i \rangle L_i(x).$$

- <u>Piecewise Polynomials in $\mathbb{R}^1$</u>

  spline, finite element, neural network

<u>key feature</u> there exists a partition of the interval $[a,b]$:

$$\Delta: \quad a \leq x_0 < x_1 < \cdots < x_n \leq b$$

such that $p(x)\Big|_{[x_i, x_{i+1}]}$ is a polynomial for $i = 0, 1, \cdots, n-1$.

(a) <u>spline</u> (fixed knotes)

$$S_m^k(\Delta) = \left\{ v(x) \in C^k(a,b) \;\middle|\; v(x)\Big|_{[x_i, x_{i+1}]} \in \mathcal{P}_m \right\}.$$

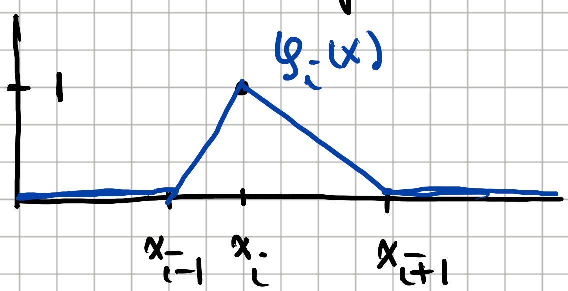
(b) finite element (fixed mesh)

C^0-element $\quad S_1^0(\Delta) = \left\{ v(x) \in C^0(a,b) \;\middle|\; v\big|_{[x_i, x_{i+1}]} \in \mathcal{P}_1 \right\}$

dimension $\quad \dim S_1^0(\Delta) = \underbrace{2}_{\dim \mathcal{P}_1} \underbrace{n}_{\#\text{ of intervals}} - \underbrace{(n-1)}_{\#\text{ of continuity constraints}} = n+1$

degrees of freedom $\quad \{\text{nodal value at } x_i\}_{i=0}^{n}$

nodal basis function $\quad \varphi_i(x_j) = \delta_{ij} = \begin{cases} 1, & j = i \\ 0, & j \neq i \end{cases}$



$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in (x_{i-1}, x_i) \\[2ex] \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x \in (x_i, x_{i+1}) \\[2ex] 0, & \text{otherwise}. \end{cases}$$

$$= \varphi_i(x;\, x_{i-1}, x_i, x_{i+1})$$

$$\forall\, v \in S_1^0(\Delta), \quad v(x) = \sum_{i=0}^{n} v(x_i)\, \varphi_i(x).$$

__error estimate__ given a function $u(x)$ defined on $[a,b]=I$

let $u_I(x)=\sum_{i=0}^{n} u(x_i)\varphi_i(x)$, then

$$\|u(x)-u_I(x)\|_{\infty,[a,b]} \le \frac{1}{8}h^2 \|u''(x)\|_{\infty,I}, \text{ where } h=\max_{1\le i\le n} h_i$$

$$h_i = x_i - x_{i-1}$$

__Proof__ $\forall x\in[x_{i-1},x_i]=I_i$, $\exists \xi_i\in(x_{i-1},x_i)$ s.t.

$$u(x)-u_I(x)=\frac{u''(\xi_i)}{2}(x-x_{i-1})(x-x_i)$$

$$\Longrightarrow \max_{x\in I_i}|u(x)-u_I(x)| \le \frac{1}{2}\|u''\|_{\infty,I_i}\max_{x\in I_i}|(x-x_{i-1})(x-x_i)|$$

$$=\frac{h_i^2}{8}\|u''\|_{\infty,I_i}$$

$$\Longrightarrow \|u-u_I\|_{\infty,I} \le \frac{h^2}{8}\|u''\|_{\infty,I}.$$

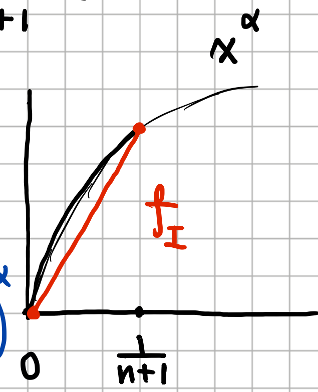
(c) __free knot spline__ (1981 Schumaker book "Spline Functions: Basic Theory")

$$\mathcal{S}_{n,1}=\left\{ s(x)\in C^0[a,b] \,\middle|\, s(x) \text{ is a piecewise linear with } n \text{ knots in } [a,b] \right\}$$

$$d(f,\mathcal{S}_{n,1})_p = \inf_{s\in\mathcal{S}_{n,1}} \|f-s\|_{L^p(I)}$$

Example $f_\alpha(x) = x^\alpha$ defined on $I=[0,1]$.

- uniform partition $\Delta_n: \ 0 < \frac{1}{n+1} < \frac{2}{n+1} < \cdots < \frac{n}{n+1} < 1$



$$d\left(f_\alpha, S^1_{n+1}(\Delta_n)\right)_\infty = \frac{1}{2}\left(\frac{1}{n+1}\right)^{\alpha}$$

$$\max_{x\in(0,\frac{1}{n+1})}\left| x^\alpha - \left(\frac{1}{n+1}\right)^{\alpha-1} x\right| = \left| x^\alpha - \left(\frac{1}{n+1}\right)^{\alpha-1} x\right|_{x=\alpha^{\frac{1}{1-\alpha}}\frac{1}{n+1}} = \alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\left(\frac{1}{n+1}\right)^{\alpha}$$

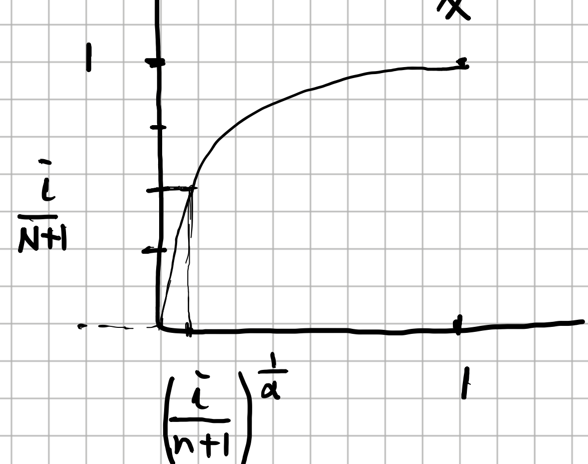
- non-uniform partition $\Delta^*: \ 0<\left(\frac{1}{n+1}\right)^{\frac{1}{\alpha}} < \cdots < \left(\frac{n}{n+1}\right)^{\frac{1}{\alpha}} < 1$

$$d\left(f_\alpha, S_{n+1,1}\right)_\infty = \frac{1}{2}\left(\frac{1}{n+1}\right)^{1}.$$

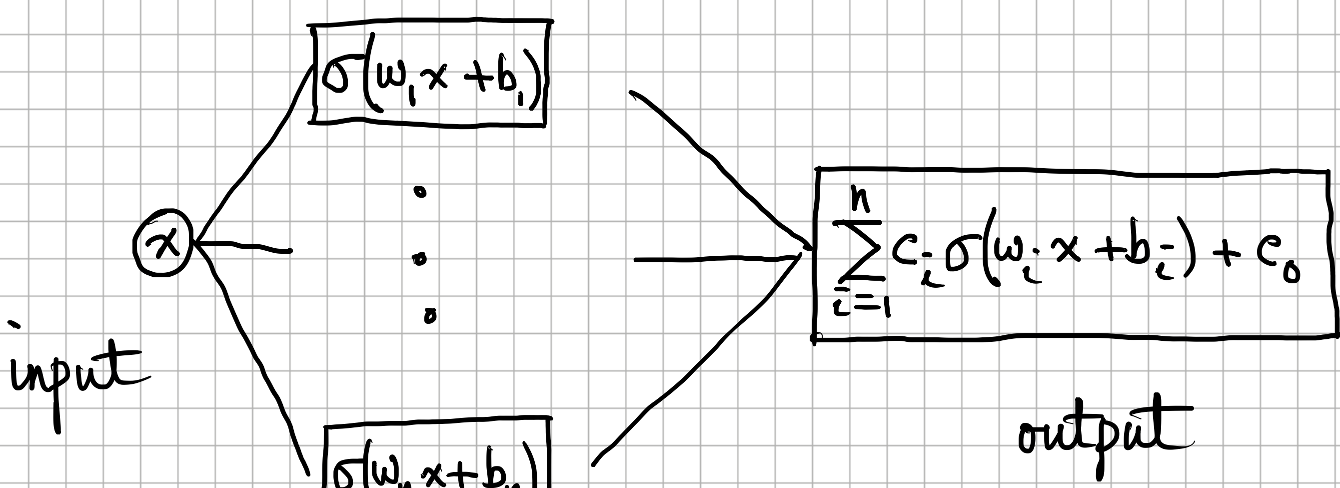
$$\max_{x\in\left(0,\left(\frac{1}{n+1}\right)^{\frac{1}{\alpha}}\right)}\left| x^\alpha - \left[\left(\frac{1}{n+1}\right)^{\frac{1}{\alpha}}\right]^{\alpha}\cdot\frac{x}{\left(\frac{1}{n+1}\right)^{1/\alpha}}\right| = \left| x^\alpha - (n+1)^{\frac{1-\alpha}{\alpha}} x\right|_{x=\alpha^{\frac{1}{1-\alpha}}\left(\frac{1}{n+1}\right)^{\alpha}}$$

$$= \frac{1}{n+1}\left|\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right| = \alpha^{\frac{1}{1-\alpha}}\left|\alpha^{-1}-1\right|\left(\frac{1}{n+1}\right)^{1}$$

why choosing Δ^*?



(d) neural network



1st – hidden layer

$\{w_i\}_{i=1}^{n}$ — weights

$\{b_i\}_{i=1}^{n}$ — bias

$$\mathcal{M}_n = \left\{ c_0 + \sum_{i=1}^{n} c_i \sigma(w_i x + b_i) \;\middle|\; c_i, w_i, b_i \in R \right\}$$

activation (ridge) function

- spline (ReLU = $k=1$)

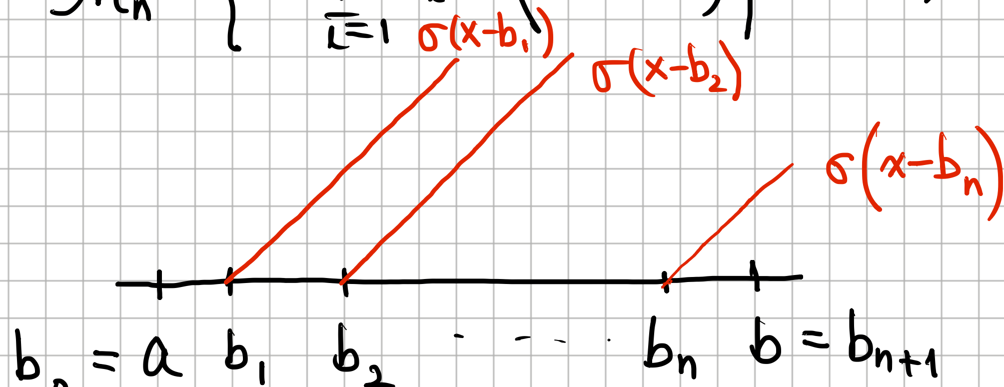
$$\sigma_k(t) = \begin{cases} 0, & t < 0 \\ t^k, & t \geq 0 \end{cases}$$

- Sigmoid

$$\sigma(t) = \frac{1}{1+e^{-t}}$$

- <u>ReLU Neural Network</u>

$$\mathcal{M}_n = \left\{ c_0 + \sum_{i=1}^{n} c_i \sigma(x - b_i) \,\middle|\, c_i \in R,\ b_i \in R \right\}$$



$\forall v \in \mathcal{M}_n$
$$\begin{cases} \bullet\ v(x) \in C^0[a,b] \\ \bullet\ v|_{[b_0, b_1]} \in \mathcal{P}_0 \\ \bullet\ v|_{[b_i, b_{i+1}]} \in \mathcal{P}_1 \quad \text{for } i = 1, \cdots, n. \end{cases}$$

(e) Relation between $\mathcal{S}_{n,1}$ and ReLU NN

$$\mathcal{S}_{n,1} = \left\{ c_{-1} + c_0 x + \sum_{i=1}^{n} c_i \sigma(x - b_i) \,\middle|\, c_i, b_i \in R \right\}.$$